The Golden Digits International Contest
6th Edition, November 2024

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for any real numbers x, y with $y \neq 0$ we have:

$$f(f(x)+y)f\left(\frac{1}{y}\right) = xf\left(\frac{1}{y}\right) + 1$$

Problem 2. Vadim and Marian play a game. Starting with Vadim, they take turns eliminating exactly one edge from a complete graph with 2024 vertices. The first player to make a move that leaves no cycles loses. Determine who has a winning strategy.

Note: A cycle is a sequence of pairwise distinct vertices $v_1v_2\dots v_n$ such that v_iv_{i+1} is an edge in the graph for evey natural number i, where idices are considere modulo n.

Problem 3. Prove that there exist infinitely many d such that we can find a polynomial P of degree d with integer coefficients and $N \in \mathbb{N}$ such that for all integers $x > N$ and any prime p we have:

$$v_p(P(x)^3 + 3P(x)^2 - 3) < \frac{d \cdot \log(x)}{2024^{2024}}$$

where $\log(x)$ denotes the natural logarithm and $v_p(n)$ denotes the largest number k such that $p^k \mid n$.

Time allowed: 4 hours and 30 minutes
Each problem is worth 7 points