The Golden Digits International Contest

6th Edition, November 2024

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for any real numbers x, y with $y \neq 0$ we have:

$$f(f(x)+y)f\left(\frac{1}{y}\right) = xf\left(\frac{1}{y}\right)+1$$

Marius Cerlat

Solution: Quite clearly, $f(\mathbb{R}^*) \neq \{0\}$, so there is some $y \in \mathbb{R}^*$ with $f\left(\frac{1}{y}\right) \neq 0$. Substituting this y in the hypothesis yields that f is injective.

Again, for some $y \in \mathbb{R}^*$ such that $f\left(\frac{1}{y}\right) \neq 0$, take $x = -\frac{1}{f\left(\frac{1}{y}\right)}$. This gives

$$f\left(y+f\left(-\frac{1}{f\left(\frac{1}{y}\right)}\right)\right) = 0$$

Therefore, there is some t with $f(t) = 0$. Putting $x = t$ in the hypothesis will return

$$f(y)f\left(\frac{1}{y}\right) = tf\left(\frac{1}{y}\right)+1$$

and since f is injective, swapping y with $1/y$ will give $t = 0$ and $f(y)f\left(\frac{1}{y}\right) = 1$.

Claim: $f(x+y) = f(x)+f(y)$

Proof: Multiplying the hypothesis by $f(y)$ gives:

$$f(f(x)+y) = x+f(y) \tag{1}$$

Setting $y \to f(y)$ in (1) yields $f(f(x)+f(y)) = x+f(f(y)) = f(f(x))+y$, for all nonzero x, and y. Thus, there is a constant c such that $f(f(x)) = x+c$ for all $x \neq 0$. Setting $x = -c$ in this relation gives $f(f(-c)) = 0 \Rightarrow c = 0 \Rightarrow f(f(x)) = x$. Now, setting $x \to f(x)$ in (1) gives the desired conclusion.

Notice that for any $x \in \mathbb{R}$ with $|x| \geq 2$, there is some $y \in \mathbb{R}^*$ s.t. $x = y+\frac{1}{y}$. Thus, by AM-GM,

$$|f(x)| = \left|f\left(y+\frac{1}{y}\right)\right| = \left|f(y)+f\left(\frac{1}{y}\right)\right| \geq 2.$$

Then, if $|y| \leq \frac{1}{2}$ $(y \neq 0)$, $|f(y)| = \frac{1}{\left|f\left(\frac{1}{y}\right)\right|} \leq \frac{1}{2}$, so f is bounded on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, implying that f is linear. A quick check gives $f(x) = x$ and $f(x) = -x$ as the only solutions.

Problem 2. Vadim and Marian play a game. Starting with Vadim, they take turns eliminating exactly one edge from a complete graph with 2024 vertices. The first player to make a move that leaves no cycles loses. Determine who has a winning strategy.

Pavel Ciurea

Solution: We show that Marian wins.

After Marian moves, the number of edges left uncut is even, so for him to lose the last cycle must be an odd one.

$t := 2024/4$. It is well known that if the number of uncut edges is $\geq 4t$, there is at least one cycle left. So, if Marian can make the graph bipartite before there are $< 4t$ edges left he wins. We show that he can do so.

Marian splits the vertices in two sets, V_1 and V_2, with $|V_1| = |V_2| = 2t$. There are $2\binom{2t}{2}$ edges uv with u, $v \in V_i$ with $i \in \{1, 2\}$ (call such an edge *good*), and before there are $< 4t$ edges left uncut, Marian cuts $\left(\binom{4t}{2} - 4t\right)/2 = 4t^2 - 3t$ edges. Thus, it is enough for him to choose the sets such that Vadim cuts at least $2\binom{2t}{2} - (4t^2 - 3t) = t$ *good* edges and he only cuts *good* ones.

At a certain point, we say a vertex u is *used* if there is an edge uv that has been cut.

Lemma: Marian can play so that after $2k - 1$ turns there are at most $2k + 1$ *used* vertices.

Proof: We show the statement by induction. The statement trivially holds for $k = 2$. We suppose the statement holds for k and show that it holds for $k + 1$.

Between the *used* vertices there are $\binom{2k+1}{2} > 2k-1$ edges, so Marian can cut an edge between two already *used* vertices (if there are $u < 2k + 1$ *used* vertices, then we consider $2k + 1 - u$ other vertices to be *used*), and since at a turn a player can make at most two new vertices *used* the statement is also true for $k + 1$ so the lemma is proven.

Therefore, after $2t - 3$ turns there are at most $2t - 1$ *used* vertices. Marian adds all these vertices to V_1, then adds random vertices until $|V_1|$ is $2t - 1$ and cuts an edge between two vertices in V_1. On the next turn Vadim cuts an edge uv. If $u \in V_1$ then Marian adds v in V_1 (if $v \in V_1$ he adds a random vertex), and if u, $v \notin V_1$ he adds u and v to V_2 and adds a random vertex to V_1.

This way, after $2t - 1$ turns Vadim cut t *good* edges and Marian cut only *good* edges, so the constructed sets satisfy the desired conditions and thus Marian wins.

Problem 3. Prove that there exist infinitely many d such that we can find a polynomial P of degree d with integer coefficients and $N \in \mathbb{N}$ such that for all integers $x > N$ and any prime p we have:

$$v_p(P(x)^3 + 3P(x)^2 - 3) < \frac{d \cdot \log(x)}{2024^{2024}}$$

where $\log(x)$ denotes the natural logarithm and $v_p(n)$ denotes the largest number k such that $p^k \mid n$.

Marius Cerlat

Solution: Let $Q(x) = x^3 + 3x^2 - 3$, and define $Q(x-1) = x^3 - 3x - 1$. Inspired by the identity

$$\left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right) = a^3 + \frac{1}{a^3},$$

we find that

$$Q\left(x + \frac{1}{x} - 1\right) = x^3 + \frac{1}{x^3} - 1.$$

Now, define $P_n(x)$ as the polynomial with integer coefficients that satisfies

$$P_n\left(x + \frac{1}{x} - 1\right) = x^n + \frac{1}{x^n} - 1.$$

Its existence and uniqueness can be easily proven by induction. We can observe that

$$P_n(P_m(x + \frac{1}{x} - 1)) = x^{mn} + \frac{1}{x^{mn}} - 1 = P_m(P_n(x + \frac{1}{x} - 1)).$$

Now, by substituting $P_n(x)$ instead of $P(x)$ in the initial condition, we find that our polynomial satisfies

$$P_3(P_n(x)) = P_n(P_3(x)).$$

Claim: For any positive constant C, there are infinitely many natural numbers n such that the degree of any irreducible polynomial dividing $P_n(x)$ is at most Cn.

Subclaim 1: For any positive constant C, there are infinitely many natural numbers n such that the degree of any irreducible polynomial dividing $x^{2n} - x^n + 1$ is at most Cn.

Proof: The polynomial $x^{2n} - x^n + 1$ divides $x^{3n} + 1$, which divides

$$x^{6n} - 1 = \prod_{d \mid 6n} \Phi_d(x),$$

where $\Phi_d(x)$ is the d-th cyclotomic polynomial. It is well-known that $\deg(\Phi_d) = \varphi(d)$, where φ is Euler's totient function. Hence, for any divisor d of $6n$,

$$\frac{\deg(\Phi_d)}{n} \leq \frac{\varphi(6n)}{n} = 6 \prod_{p_i \mid 6n} \left(\frac{p_i - 1}{p_i}\right),$$

where p_i are the prime factors of $6n$. It is known that

$$\prod \left(\frac{p-1}{p} \right)$$

converges to 0. Therefore, by taking n to be divisible by primes $p_1, p_2, \ldots, p_k$ such that

$$\prod \frac{p_i - 1}{p_i} < \frac{C}{6},$$

the claim is proved.

Subclaim 2: If a polynomial $Q(x)$ is irreducible, then $Q_R(x) = x^{\deg(Q)} Q\left(x + \frac{1}{x}\right)$ has at most two irreducible factors.

Proof: Assume $Q_R(x) = M(x)N(x)$, where $M(x)$ is an irreducible polynomial. Our goal is to prove that $N(x)$ must also be irreducible. We have

$$M(x)N(x) = x^{\deg(Q)} Q\left(x + \frac{1}{x}\right) = x^{2\deg(Q)} M\left(\frac{1}{x}\right) N\left(\frac{1}{x}\right).$$

This implies that the polynomial

$$\bar{M}(x) = x^{\deg(M)} M\left(\frac{1}{x}\right)$$

is irreducible. Now, we consider two cases:

1. If $\bar{M}(x)$ divides $M(x)$, then since both polynomials have the same degree, we conclude that $M(x)$ is simply a scaled version of $\bar{M}(x)$. If $M(1)$ is non-zero, then since $\bar{M}(1) = M(1)$, we have that the polynomials are equal. If the degree of $M(x)$ is even, $\frac{M(x)}{x^{\deg(M)/2}}$ is a polynomial in $x + \frac{1}{x}$, contradicting the hypothesis that $Q(x)$ is irreducible. When $deg(M)$ is odd, we observe that $M(-1) = 0$, meaning that $Q_R(-1) = 0$, and thus that $Q(-1 - \frac{1}{1}) = 0$, meaning Q is of the form $k(x+2)$, for which the hypotesis is clearly true.

If instead $M(1) = 0$, since $M(x)$ divides $Q_R(x)$ and $Q_R(1) = 0$, we have $Q(1 + \frac{1}{1}) = 0$, and thus $Q(x) = c(x-2)$, which satisfies the condition.

2. If $\bar{M}(x)$ divides $N(x)$, and if $\frac{N(x)}{M(x)}$ has degree at least 1, we observe that $M(x)M\left(\frac{1}{x}\right)$ is symmetric in $\frac{1}{x}$, which again leads to a contradiction with the assumption that $Q(x)$ is irreducible. If no contradiction arises, then $Q_R(x)$ must have exactly two irreducible factors, as needed.

Finally, combining these two facts, we consider an irreducible factor $F(x)$ of $P_n(x)$, where n is as described in the first subclaim. We conclude that

$$x^{\deg(F)} F\left(x + \frac{1}{x}\right)$$

has degree $2\deg(F)$ and at most two prime factors of degrees at most Cn, completing the proof of the claim.

Now, consider n such that all irreducible factors of $P_n(x)$ have degree at most $\frac{n}{2024^{2025}}$. We will show that such n satisfy the condition. Indeed, take an irreducible factor $F(x)$ of the polynomial, and assume $p \mid F(k)$ for a natural number k. By Bézout's Lemma, for any two coprime polynomial factors $F_1(x)$ and $F_2(x)$ of P_n (since they are irreducible and distinct), there exist polynomials $A(x)$ and $B(x)$ with integer coefficients such that

$$F_1(x)A(x) + F_2(x)B(x) = c,$$

where c is a constant. Here, it is important to notice that $P_n(x)$ does not have repeated factors, as that would imply $x^{6n} - 1$ also having double factors. Doing this for every pair of irreducible divisors, we obtain the existence of a constant M (independent of p) such that for any k,

$$v_p(P_n) \leq v_p(F(k)) + M.$$

For an irreducible factor $F(x)$, this is at most $\log_p(F(k)) + M$. Using the fact that for large enough x, $F(x) < x^{\frac{n}{2 \cdot 2024^{2024}}}$, and that $\log_p(x) \leq \log_2(x)$, we arrive at the desired conclusion.