The Golden Digits International Contest

8th Edition, January 2025

Problem 1. Alex and Bob play a game: Bob picks an initial positive integer x_0. Then, after every minute, Alex chooses a positive integer a, and Bob chooses x_{i+1} to be equal to $x_i + a$ or $x_i + 2a$. Prove that no matter the choice of x_0 and Bob's strategy, Alex can force him to choose a number that is a perfect square after a finite number of minutes.

David Anghel

Solution: Let $k \in \mathbb{N}$ such that $x_0 < 2k^2 - 1$. Then by choosing $a = 1$ at every turn Alex can guarantee that there exists an index i such that $x_i = 2k^2 - 1$ or $x_i = 2k^2$.

If $x_i = 2k^2 - 1$ then Alex chooses $a = 7k^2 + 12k + 5$, and x_{i+1} is either $(3k + 2)^2$ or $(4k + 3)^2$ so Alex wins.

If $x_i = 2k^2$ then Alex chooses $a = 7k^2$, and then x_{i+1} is either $(3k)^2$ or $(4k)^2$ so Alex wins in this case as well, so our proof ends.

Problem 2. Let n, m be two integers such that $2 \mid mn$. On an $n \times m$ board we place $n \cdot m/2$ dominoes without overlap. On some domino e lies a burito. Alex sits on the top-left corner of a domino s and is very hungry. He is allowed to make two types of moves:

a) from the vertex of a domino he can move diagonally to the opposite one

b) if he sits on the corner of some domino d he can move to the top-left corner of d

Alex can eat the burito if he reaches a corner of e. Can Alex satisfy his belly regardless of the choice of m, n, s, and e?

Pavel Ciurea

Solution: We prove the answer is yes.

Let $G = (V, E)$ be a directed graph in which each vertex encodes a domino. We draw an edge $u \to v$ if their corresponding dominoes share at least a vertex and one of them is either the top-left or the bottom-right vertex of u. We show that G is strongly connected (i.e. there is a directed path from u to v, for any u, $v \in V$).

Suppose for the sake of contradiction that G is not strongly connected, Then G can be split into strongly connected components C_1, C_2, $\ldots C_k$ where k is minimal. We create a new directed graph $G_0 = (E_0, V_0)$ in which each vertex v_i corresponds to a component C_i, and we draw an edge $v_i \to v_j$ if there exists $u \in C_i$ and $v \in C_j$ such that $\vec{uv} \in E$.

Claim: There exists i such that $deg_+ v_i = 0$.

Proof: We show that if $deg_+ v_i \geq 1$ for all $i = \overline{1, k}$ then we can find a directed cycle in G_0.

We create a list $v_1 = v_{i_1}, v_{i_2}, v_{i_3}, \ldots$ such that $\overrightarrow{v_{i_t} v_{i_{t+1}}} \in E_0$. Since we have a finite number of edges at some point we will find a vertex that repeats in the list, say $v_{i_t} = v_{i_{t+\delta}}$. Then $v_{i_t} \ldots v_{i_{t+\delta-1}}$ is a directed cycle in G_0.

But then the union of the connected components corresponding to the vertices in the cycle is a strongly connected component, contradiction with the choice of k.

Without loss of generality v_1 has outdegree 0.

Call a cell neighbour of a vertex v (vertex of a domino) if one of its vertices is v. We say a cell has colour i if it is covered by a domino in C_i.

Claim: Let d be a domino in C_1. Then the neighbouring cells of d's top-left and bottom-right corners have colour 1.

Proof: Suppose the claim is false. We treat the case where the the top-left corner (tl) of d has a neighbour of a different colour, as the other case is analogous. Call the domino which covers a neighbouring cell of tl and doesn't have colour 1 d_2.

tl cannot be a vertex of d_2 as otherwise $\vec{v_1 v_2} \in E_0$. Then d_2 must cover two neighbouring cells of tl. Call the one not covered by either d_2 or d B. Suppose that B is the top-right neighbour of tl. Call such cells “bad”. Without loss of generality B is one of the “bad” cells with maximal y-coordinate (i.e. B is the highest “bad” cell).

B has colour 1 because the domino covering it (d’) has tl as a vertex. If this domino is horizontal its top-left corner is a corner of d_2, contradiction ($\overrightarrow{v_1 v_i} \in E$ where i is the colour of d_2). If it is vertical domino, then the bottom-left corner of its top-left corner cannot have colour 1, because it would imply $\overrightarrow{v_1 v_i} \in E$. Then the domino covering it cannot be horizontal because it would have the top-left corner of d’ as a vertex, so it must be vertical. Then the top-right neighbour of the top-left corner of d’ is a “bad” vertex, contradiction with the choice of B.

The case where B is the bottom-left neighbour of tl is analogous.

Since G is not strongly connected, the dominoes corresponding to the vertices in C_1 cannot cover the entire board, so there must be a domino d' which shares at least one point on its border with a domino d corresponding to a vertex in C_1.

Suppose d is horizontal. If d' covers the top-left neighbour of the top-right vertex of d then the bottom-left corner of d' is the bottom-right corner of the domino covering the top-right neighbour of the top-left vertex of d. But this domino corresponds to C_1 from the claim, contradiction.

This proves that the top-left neighbour of the top-right vertex of d has colour 1, so the top-right corner of d is the bottom-right corner of another domino corresponding to a vertex in C_1. So, from the claim, d' cannot cover the top-right neighbour of the top-right corner of d.

So we are left to check the cases when d' covers the bottom-right or the bottom-left neighbours of the bottom-left corner of d, which are analogous to the previous case.

The case where d is placed vertically is treated in the same way.

So, the dominoes corresponding to C_1 must cover the entire board, so G is strongly connected, ending our proof.

Problem 3. Let $\mathcal{P}$ and $\mathcal{Q}$ be convex polygons with areas $S_{\mathcal{P}}$ and $S_{\mathcal{Q}}$, respectively, such that every vertex of $\mathcal{Q}$ lies inside or on the boundary of $\mathcal{P}$. Prove that there exists a polygon $\mathcal{R}$, similar to $\mathcal{P}$ and with sides parallel to the sides of $\mathcal{P}$, with area $S_{\mathcal{R}}$, such that every vertex of $\mathcal{R}$ lies inside or on the boundary of $\mathcal{Q}$, and

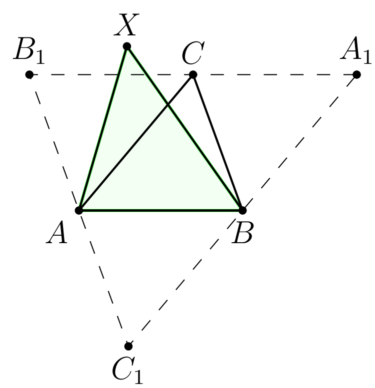
$$S_{\mathcal{R}} \geq \frac{1}{1000} \cdot \frac{S_{\mathcal{Q}}^2}{S_{\mathcal{P}}}$$

Anghel David

Solution: We will prove the following key lemma.

Lemma 1. If $\mathcal{T}$ is a convex polygon with area $S_{\mathcal{T}}$, then there exists a triangle Δ with area S_{Δ} and vertices inside or on the boundary of $\mathcal{T}$ such that $S_{\Delta} \geq \frac{1}{4} S_{\mathcal{T}}$.

Proof: Let us consider $\triangle ABC$ with vertices being vertices of $\mathcal{T}$ and maximal area. Construct lines ℓ_A, ℓ_B, ℓ_C through the respective vertices, parallel to the lines $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$. Let $\ell_A \cap \ell_B = C_1$, $\ell_B \cap \ell_C = A_1$, and $\ell_C \cap \ell_A = B_1$. If we assume that there exists a point X of $\mathcal{T}$ outside $\triangle A_1B_1C_1$, we easily reach a contradiction with the maximality of $\triangle ABC$. Therefore, $\mathcal{T}$ lies entirely within $\triangle A_1B_1C_1$, from which it follows that $S_{\Delta} \geq \frac{1}{4} S_{\mathcal{T}}$.

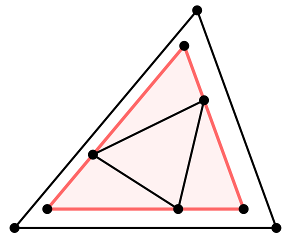


From the lemma, it follows that there exists a triangle $\Delta_{\mathcal{Q}} \subset \mathcal{Q}$ with area $S_{\Delta_{\mathcal{Q}}} \geq \frac{1}{4} S_{\mathcal{Q}}$. Additionally, considering the triangle ABC in $\mathcal{P}$ with the largest area, the triangle $A_1B_1C_1$, defined as in the lemma, contains $\mathcal{P}$ entirely and has area $S_{\triangle ABC} \leq 4 S_{\mathcal{P}}$. For brevity, let us denote $\Delta_{\mathcal{P}} = \triangle A_1B_1C_1$. Now we will prove a lemma that completes the problem.

Lemma 2. There exists a triangle $\Delta_{\mathcal{R}}$, similar to $\Delta_{\mathcal{P}}$, whose vertices lie inside or on the boundary of $\Delta_{\mathcal{Q}}$, with area

$$S_{\Delta_{\mathcal{R}}} \geq \frac{S_{\Delta_{\mathcal{Q}}}^2}{S_{\Delta_{\mathcal{P}}}}.$$

Proof: Note that $\Delta_{\mathcal{Q}} \subset \mathcal{Q} \subset \mathcal{P} \subset \Delta_{\mathcal{P}}$, meaning $\Delta_{\mathcal{Q}}$ is entirely contained within $\Delta_{\mathcal{P}}$, which implies $S_{\Delta_{\mathcal{Q}}} < S_{\Delta_{\mathcal{P}}}$. If one of the sides of $\Delta_{\mathcal{P}}$ does not contain a vertex of $\Delta_{\mathcal{Q}}$, we can fix the other two sides and translate the third side parallelly toward $\Delta_{\mathcal{Q}}$ until a vertex of $\Delta_{\mathcal{Q}}$ lies on the third side.



In this way, we obtain a triangle $\Delta'_{\mathcal{P}} \sim \Delta_{\mathcal{P}}$ with an area smaller than that of $\Delta_{\mathcal{P}}$. Therefore, if we prove the lemma with $\Delta_{\mathcal{P}}$ replaced by $\Delta'_{\mathcal{P}}$, the inequality for $\Delta_{\mathcal{P}}$ will follow trivially. Thus, we can assume that every side of $\Delta_{\mathcal{P}}$ contains a vertex of $\Delta_{\mathcal{Q}}$. Now, we consider several different cases regarding the placement of these vertices on the sides of $\Delta_{\mathcal{P}}$.

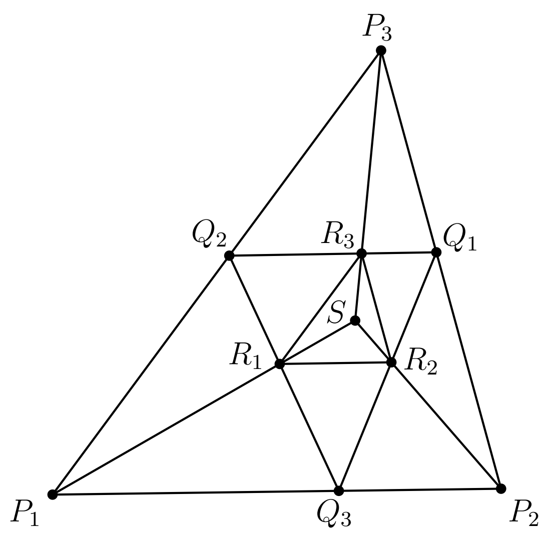
Case 1: $\Delta_{\mathcal{P}}$ and $\Delta_{\mathcal{Q}}$ have no common vertices.

In this case, each side of $\Delta_{\mathcal{P}}$ must contain a vertex of $\Delta_{\mathcal{Q}}$ in its interior. Let $\Delta_{\mathcal{P}} = \triangle P_1P_2P_3$ and $\Delta_{\mathcal{Q}} = \triangle Q_1Q_2Q_3$, with $Q_i \in P_{i+1}P_{i+2}$, where the indices are taken modulo 3. Let S be a point in the interior of $\Delta_{\mathcal{Q}}$ such that

$$S_{\triangle Q_1SQ_2} : S_{\triangle Q_2SQ_3} : S_{\triangle Q_3SQ_1} = S_{\triangle Q_1P_3Q_2} : S_{\triangle Q_2P_1Q_3} : S_{\triangle Q_3P_2Q_1}.$$

Such a point exists because we can choose S so that

$$\mathrm{dist}(S, Q_1Q_2) : \mathrm{dist}(S, Q_2Q_3) : \mathrm{dist}(S, Q_3Q_1) = \frac{S_{\triangle Q_1P_3Q_2}}{Q_1Q_2} : \frac{S_{\triangle Q_2P_1Q_3}}{Q_2Q_3} : \frac{S_{\triangle Q_3P_2Q_1}}{Q_3Q_1}.$$



Now let $R_i = SP_i \cap Q_{i+1}Q_{i+2}$. We can observe that

$$\frac{SR_3}{R_3P_3} : \frac{SR_1}{R_1P_1} : \frac{SR_2}{R_2P_2} = \frac{S_{\triangle Q_1SQ_2}}{S_{\triangle Q_1P_3Q_2}} : \frac{S_{\triangle Q_2SQ_3}}{S_{\triangle Q_2P_1Q_3}} : \frac{S_{\triangle Q_3SQ_1}}{S_{Q_3P_2Q_1}} = 1 : 1 : 1,$$

from which it follows that $\triangle R_1R_2R_3 \sim \Delta_{\mathcal{P}}$ since $R_iR_{i+1} \parallel P_iP_{i+1}$. Let $\Delta_{\mathcal{R}} = \triangle R_1R_2R_3$. It remains to observe that if we denote $\lambda = \frac{R_iP_i}{SR_i}$, then we have:

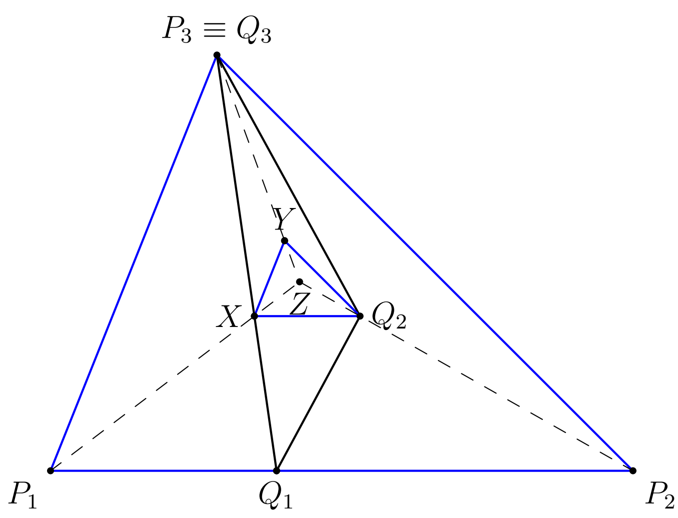
$$\begin{aligned}\frac{S_{\Delta_{\mathcal{Q}}}}{S_{\Delta_{\mathcal{R}}}} &= 1 + \frac{S_{\triangle R_1Q_2R_3} + S_{\triangle R_3Q_1R_2} + S_{\triangle R_2Q_3R_1}}{S_{\triangle R_1SR_3} + S_{\triangle R_3SR_2} + S_{\triangle R_2SR_1}} \\ &= 1 + \frac{\lambda S_{\triangle R_1SR_3} + \lambda S_{\triangle R_3SR_2} + \lambda S_{\triangle R_2SR_1}}{S_{\triangle R_1SR_3} + S_{\triangle R_3SR_2} + S_{\triangle R_2SR_1}} \\ &= 1 + \lambda.\end{aligned}$$

From the homothety, however, $S_{\Delta_P} = (1+\lambda)^2 S_{\Delta_R}$, hence $S_{\Delta_R} = \frac{S^2_{\Delta_Q}}{S_{\Delta_P}}$, completing the proof.

Case 2: One of the vertices of $\Delta_{\mathcal{P}}$ coincides with one of the vertices of $\Delta_{\mathcal{Q}}$.

Let the triangles be $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$, similar to the previous case, with $P_3 \equiv Q_3$. Let Q_1 lie on side P_1P_2, and let Q_2 lie in the interior or on the perimeter of $\triangle Q_3Q_1P_2$. We will construct the triangle $\Delta_{\mathcal{R}}$ explicitly:

- Construct a line ℓ through Q_2, parallel to $\overline{P_1P_2}$, and define $X = \overline{Q_1Q_3} \cap \ell$.
- Construct a point Y such that $\overline{XY} \parallel \overline{P_1P_3}$ and $\overline{YQ_2} \parallel \overline{P_2P_3}$. It is not difficult to observe that Y does not lie outside $\Delta_{\mathcal{Q}}$.
- Consider the triangle $\Delta_{\mathcal{R}} = \triangle XYQ_2 \subset \Delta_{\mathcal{Q}}$. Since $\Delta_{\mathcal{R}} \sim \triangle P_1P_2P_3$ and the two triangles have pairwise parallel sides, the lines $\overline{UP_1}$, $\overline{P_2Q_2}$, and $\overline{VP_3}$ intersect at a single point. Let this intersection point be Z.



It is easy to observe that if $k = \frac{ZX}{ZP_1} = \frac{ZQ_2}{ZP_2} = \frac{ZY}{ZP_3}$, then:

$$\begin{aligned}S_{\triangle XYZ} : S_{\triangle XZP_3} : S_{\triangle ZP_3P_1} &= S_{\triangle XZQ_2} : S_{\triangle XZQ_2Q_1} : S_{\triangle ZP_1P_2} \\ &= S_{\triangle YZQ_2} : S_{\triangle ZQ_2P_3} : S_{\triangle ZP_2P_3},\end{aligned}$$

because all three ratios are $1 : k : k^2$. Summing up the areas, we obtain:

$$S_{\Delta_{\mathcal{R}}} : S_{\Delta_{\mathcal{Q}}} : S_{\Delta_{\mathcal{P}}} = 1 : k : k^2,$$

from which $S_{\Delta_{\mathcal{R}}} = \frac{S^2_{\Delta_{\mathcal{Q}}}}{S_{\Delta_{\mathcal{P}}}}$. This concludes Case 2, and Lemma 2 is proved.

From Lemma 2, it follows that there exists a triangle $\Delta_{\mathcal{R}} \sim \Delta_{\mathcal{P}}$ such that $\Delta_{\mathcal{R}} \subset \Delta_{\mathcal{Q}}$ and its area satisfies

$$S_{\Delta_{\mathcal{R}}} \geq \frac{S^2_{\Delta_{\mathcal{Q}}}}{S_{\Delta_{\mathcal{P}}}} \geq \frac{\left(\frac{1}{4} S_{\mathcal{Q}}\right)^2}{4 S_{\mathcal{P}}} = \frac{1}{64} \cdot \frac{S^2_{\mathcal{Q}}}{S_{\mathcal{P}}}.$$

Let $\mathcal{P}'$ be the image of $\mathcal{P}$ under the homothety mapping $\Delta_{\mathcal{P}}$ to $\Delta_{\mathcal{R}}$. We can observe that $\mathcal{P}' \subset \Delta_{\mathcal{R}} \subset \Delta_{\mathcal{Q}} \subset \mathcal{Q}$, which implies that $\mathcal{P}'$ lies entirely within $\mathcal{Q}$. Moreover, $\mathcal{P}' \sim \mathcal{P}$ due to the homothety, and furthermore,

$$S_{\mathcal{P}'} = \frac{S_{\mathcal{P}}}{S_{\Delta_{\mathcal{P}}}} \cdot S_{\Delta_{\mathcal{R}}} \geq \frac{S_{\mathcal{P}}}{S_{\Delta_{\mathcal{P}}}} \cdot \left(\frac{1}{64} \cdot \frac{S^2_{\mathcal{Q}}}{S_{\mathcal{P}}} \right) = \frac{1}{64} \cdot \frac{S^2_{\mathcal{Q}}}{S_{\Delta_{\mathcal{P}}}} \geq \frac{1}{256} \cdot \frac{S^2_{\mathcal{Q}}}{S_{\mathcal{P}}}.$$

The polygon $\mathcal{P}'$ satisfies the desired conditions for $\mathcal{R}$, completing the solution to the problem.