The Golden Digits National Contest

3rd Edition, April 2024

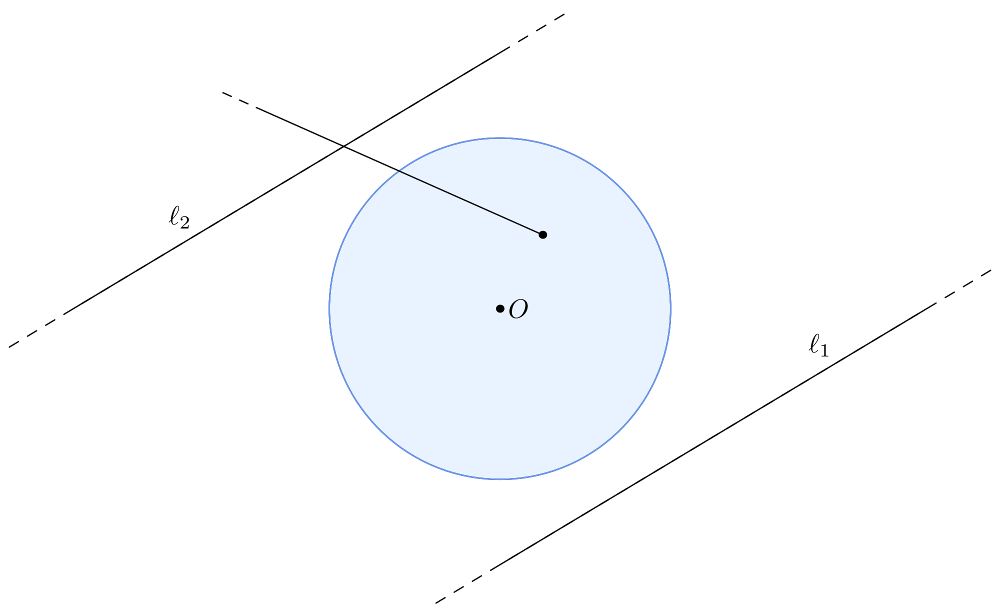
Problem 1. Vlad draws one hundred rays in the Euclidean plane. David then draws a line ℓ and pays Vlad one pound for each ray that ℓ intersects. Naturally, David wants to pay as little as possible. What is the largest amount of money that Vlad can get from David?

Vlad-Titus Spătaru

Solution. The answer is 49 pounds.

Firstly, note that Vlad can force David to pay him at least 49 pounds by combining the 100 rays into pairs, thus forming 50 non-parallel lines. David's line will be parallel to at most one of Vlad's lines and will intersect all the others: thus, ℓ intersects at least 49 rays.

Next, we will show that regardless of the way Vlad draws the 100 rays, David can draw a line ℓ which intersects at most 49 rays. To do so, fix a point O in the plane and consider a disk centered at O which is large enough that it contains all the endpoints of Vlad's rays.



If David draws two parallel lines ℓ_1 and ℓ_2 outside this disk, then any ray may intersect at most one of them, since any ray pierces the disk exactly once. Furthermore, if David draws these lines parallel to one of the Vlad's rays, it won't intersect either ℓ_1 or ℓ_2.

This leaves a total of 99 rays which can intersect at most one of ℓ_1 and ℓ_2. Therefore, by the pigeonhole principle, some ℓ_i intersects at most 49 rays, hence by drawing that line David can pay Vlad at most 49 pounds, which finishes the proof.

Problem 2. Let $\mathbb{Z}[x]$ be the set of integer polynomials. Find all the functions $\varphi : \mathbb{Z}[x] \to \mathbb{Z}[x]$ such that $\varphi(x) = x$, any integer polynomials f, g satisfy $\varphi(f+g) = \varphi(f) + \varphi(g)$, and $\varphi(f)$ is a perfect power if and only if f is a perfect power.

Pavel Ciurea

Solution. For an integer polynomial f we define the *spectrum* of f to be the set of positive integers n for which $f = g^n$ for some polynomial $g \in \mathbb{Q}[x]$, denoted by $\operatorname{Spec}(f)$.

If $\varphi(f) = 0$ and $f \neq 0$ then for any integer k we have $\varphi(kf) = 0$. As 0 is a perfect power, kf must always be a perfect power, which is absurd. Hence, $\varphi(f) = 0$ implies $f = 0$.

Let $\varphi(1) = g$. As observed previously, $g \neq 0$. Assume that $g \neq 1$. Then, we may choose a prime number p which doesn't divide any element of $\operatorname{Spec}(f)$, as this set is either finite or consists of the odd positive integers. Let α be the leading coefficient of g.

Choose a prime number $q > |\alpha|$. Then, $f(q^p) = q^p g$, so $q^p g$ is a perfect power. Write $q^p g = h^n$ for some integer polynomial h with leading coefficient β and $n \geqslant 2$. Comparing leading coefficients, $q^p \alpha = \beta^n$ so $n\nu_q(\beta) = p$. As $n \geqslant 2$ this forces $n = p$. Thus, $g = (h/q)^p$, absurd.

Therefore, $\varphi(1) = 1$. Recall that $\varphi(x) = x$. We will prove using strong induction that $\varphi(x^k) = x^k$ for every non-negative integer k. Assume that this is true for all $0 \leqslant k < n$ for some $n \geqslant 2$ (the base case $n = 2$ being already proven). We will show that $\varphi(x^n) = x^n$.

Observe that $\varphi((x+k)^n) = \varphi(x^n) - x^n + (x+k)^n$. Because $(x+k)^n$ is a perfect power, then the latter polynomial is also a perfect power, say $h_k^{n_k}$ where h_k is an integer polynomial and $n_k \geqslant 2$. For the sake of simplicity, let $g = \varphi(x^n) - x^n$.

Observe that $g = \pm 1$ is impossible, as $\varphi(x^n)$ ought to be a perfect power, but the polynomials $x^n \pm 1$ are not perfect powers, since they do not have double roots. Now, assume that $g \neq 0$ and fix $x = \lambda \in \mathbb{Z}$ such that $|g(\lambda)| > 1$. Then, $g(\lambda) + (\lambda + k)^n = h_k(\lambda)^{n_k}$ for any integer k.

In other words, there exists an integer a with $|a| > 1$ such that $a + b^n$ is a perfect power for any integer b. Let p be a prime factor of a. Let $\alpha = \nu_p(a)!$ and $b = (pc)^\alpha$. Then, $\nu_p(a + (pc)^{n\alpha}) = \nu_p(a)$ so $a + (pc)^{n\alpha} = u^v$ with $2 \leqslant v \leqslant \nu_2(a)$. Let $(pc)^{n\alpha/v} = \beta$.

Choose c large enough such that $a + (pc)^{n\alpha} > 0$. Then, we may assume that $u \geqslant 0$ as well and note that $\beta \geqslant 0$ too. Therefore, $|a| = |u^v - \beta^v| \geqslant u + \beta$. Letting c be large enough yet again, we reach a contradiction. Therefore, $g = 0$ so $\varphi(x^n) = x^n$, as desired.

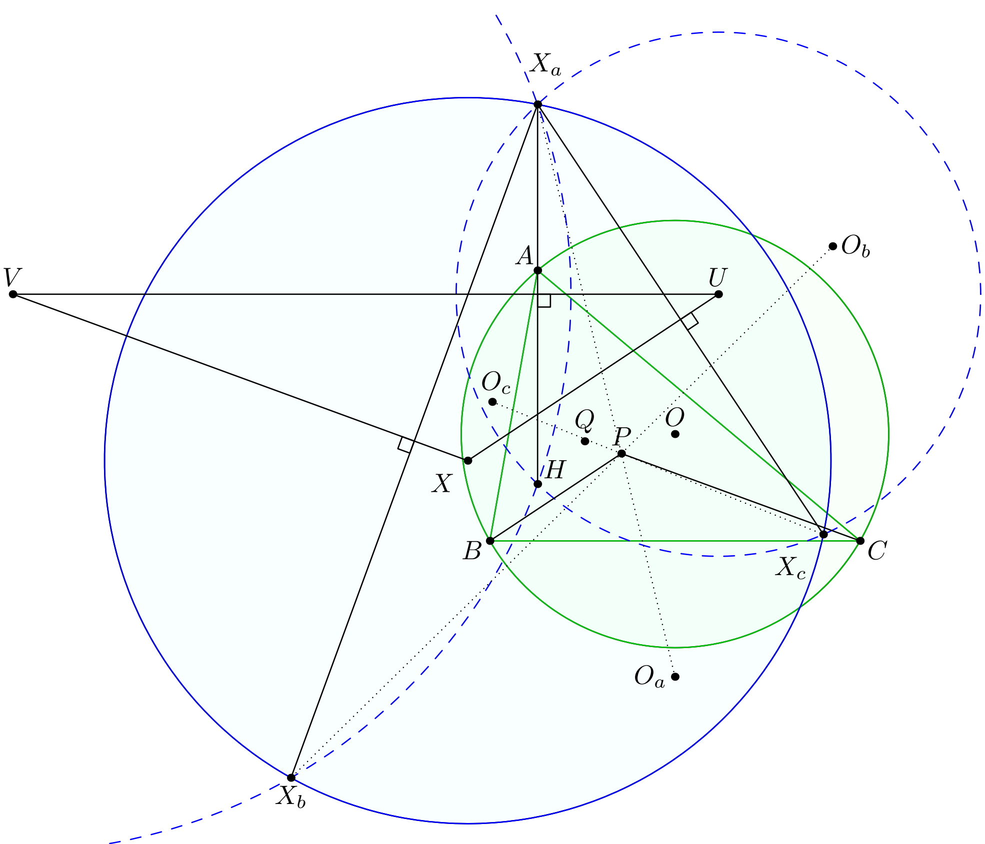
To conclude, because $\varphi(x^k) = x^k$ for any non-negative integer k, using the additivity of φ it follows that $\varphi(f) = f$ for any polynomial f. This function satisfies the given conditions trivially.

Problem 3. Let ABC be an acute scalene triangle with orthocentre H and circumcentre O. Let P be an arbitrary point on the segment OH and O_a be the circumcentre of PBC. The line PO_a intersects the line HA at X_a. Define X_b and X_c similarly. Let Q be the isogonal conjugate of P and X be the circumcentre of $X_aX_bX_c$. Prove that PQ and HX are parallel.

David-Andrei Anghel

Solution. Firstly, observe that $AH \parallel OO_a$, so by Thales' theorem, $PO_a/PX_a = PO/PH$. We get the corresponding ratios for B and C similarly: $PO_b/PX_b = PO/PH$ and $PO_c/PX_c = PO/PH$. Consequently, the triangles $X_aX_bX_c$ and $O_aO_bO_c$ are homothetic triangles.

Since $O_bO_c \perp AP$, we have $X_bX_c \perp AP$ and similarly we get $X_aX_c \perp BP$ and $X_bX_a \perp CP$. Now, let U and V be the circumcentres of HX_aX_C and HX_aX_b, respectively. Observe that that $UV \perp X_aH$ and, in turn, $X_aH \perp BC$, which yields $UV \parallel BC$.



Observe further that $XU \perp X_aX_c$ and $X_aX_c \perp PB$, so $XU \parallel BP$. Similarly, we obtain $XV \parallel CP$. Therefore, the triangles XUV and PBC are homothetic. As U is the circumcentre of HX_aX_C, using directed angles

$$\begin{aligned}\measuredangle HUV &= \measuredangle(HU, UV) = \measuredangle(HU, HX_a) + \pi/2 \\ &= \measuredangle UHX_a + \pi/2 = \measuredangle HX_cX_a = \measuredangle ABP = \measuredangle QBC,\end{aligned}$$

as $HX_c \perp AB$ and $X_cX_a \perp BP$. Similarly, we have $\measuredangle HVU = \measuredangle QCB$. Finally, as $UV \parallel BC$, this also implies that triangles QBC and HUV are homothetic. Therefore, the homothety that maps $U \mapsto B$ and $V \mapsto C$ will map $H \mapsto Q$ and $X \mapsto P$, so $PQ \parallel HX$, as desired.