The Golden Digits International Contest

7th Edition, December 2024

Problem 1. On a table, there are 2025 empty boxes numbered $1, 2, \ldots, 2025$ and 2025 balls with weights $1, 2, \ldots, 2025$. Starting with Vadim, Vadim and Marian take turns selecting a ball from the table and placing it into an empty box. After all 2025 turns, there is exactly one ball in each box. Denote the weight of the ball in box i by w_i. Marian wins if

$$\sum_{i=1}^{2025} i \cdot w_i \equiv 0 \pmod{23}$$

If both players play optimally, can Marian guarantee a win?

Ciurea Pavel

Solution: We show Marian cannot win.

We associate each move a player makes a pair (i, j), $0 \leq i, j \leq 22$, such that if the player places a ball of weight w in box b, then $w \equiv j \pmod{23}$ and $b \equiv i \pmod{23}$.

Vadim starts by making a move $(1, 1)$. After that, if Marian makes a move (i, j) Vadim makes a move $(i, -j)$. We show that Vadim can always do so:

For $0 \leq i \leq 22$, let $B_{i,M}$ and $W_{i,M}$ be the number of boxes with number $b \equiv i \pmod{23}$, and the number of balls of weight $w \equiv i \pmod{23}$ after move M, respectively.

Obviously, $B_{i,1} = W_{i,1} = 88$. So, after Vadim moves, say at move i, $B_{i,M} \equiv 0 \pmod{2}$ and $W_{i,j} = W_{i,23-j}$. Now it is easy to see that Vadim can always stick to his strategy.

Then, after 2025 moves $\sum_{i=1}^{2025} i \cdot w_i \equiv 1 \pmod{23}$. Thus, Marian cannot win.

Problem 2. Let $n \in \mathbb{N}$, $n \geq 2$. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f^2(x_1 + \cdots + x_n) = \sum_{i=1}^{n} f^2(x_i) + 2\sum_{i<j} f(x_i x_j)$$

for all $x_1, \ldots, x_n \in \mathbb{R}_{>0}$.

Vila Andrei

Solution 1: Let $P(x_1, x_2, \ldots, x_n)$ denote the assertion $f^2(x_1 + \cdots + x_n) = \sum_{i=1}^{n} f^2(x_i) + 2\sum_{i<j} f(x_i x_j)$

$f^2(x_1 + \cdots + x_n) = \sum_{i=1}^{n} f^2(x_i) + 2\sum_{i<j} f(x_i x_j) > f^2(x_1)$ so f is increasing.

Claim 1: $\forall m \in \mathbb{R}_{>0}$ there is some x such that $f(x) < m$

Proof: Suppose for the sake of contradiction that $f(x) \geq m$, $\forall x \in \mathbb{R}_{>0}$.

We take $x_2 = x_3 = \ldots x_n = \frac{y}{n-1}$. We get $f^2(x+y) \geq f^2(x) + m$.

Let k be a natural number. Then summing the inequality $f^2\left(x + \frac{y}{k}\right) > f^2\left(x + \frac{y-1}{k}\right) + m$ for $y = \overline{1,k}$ we get that $f^2(x+1) > m \cdot k$, which is obviously false for k big enough.

Claim 2: $f(x) + f(y) = f(a) + f(b)$, $\forall x, y, a, b \in \mathbb{R}_{>0}$ such that $x + y = a + b$.

Proof: Case 1: $n = 2$

$P(x+y, z)$ and $P(x, y)$ give $f(x)^2 + f(y)^2 + 2f(xy) + f(z)^2 + 2f(xz + yz) = f^2(x+y)^2 + f(z)^2 + 2f(xz + yz) = f^2(x + y + z)$

Considering the same equation, but swapping y and z, after cancelling common terms we get

$$f(xy) + f(xz + yz) = f(xz) + f(xy + zy)$$

so this case is proven.

Case 2: $n \geq 3$

Looking at $P(x + x_1, x_2, y, y, \ldots, y)$ and $P(x_1, x + x_2, y, y, \ldots, y)$ we get

$$E(x_1) - E(x_2) + 2(n-1)(f((x + x_1)y) - f(yx_1) - f((x + x_2)y) + f(yx_2)) = 0$$

where $E(x_1) = f^2(x + x_1) - f^2(x_1)$ and similarly for x_2.

Thus, there exists $c \in \mathbb{R}$ such that $f((x + x_1)y) - f(yx_1) - f((x + x_2)y) + f(yx_2) = c$, $\forall y$.

From claim 1 and because f is increasing, $\forall \varepsilon > 0$ there is some t such that $\forall x < t$, $f(x) < \varepsilon$. Thus, for every small enough y we get that $|f((x+x_1)y) - f(yx_1) - f((x+x_2)y) + f(yx_2)| < 2\varepsilon$, so $c = 0$, thus

$$f((x + x_1)y) + f(yx_2) = f(yx_1) + f((x + x_2)y)$$

so the claim is proven.

Claim: $f(x) + f(y) = f(x + y)$

Proof: We first show that $\forall x, m > 0$ there exists $\varepsilon > 0$ such that $f(x) - f(x - \varepsilon) < m$

By claim 2 $f(x) - f(x - \varepsilon) = f(2\varepsilon) - f(\varepsilon)$. By claim 1 we can simply choose ε such that $f(2\varepsilon) < m$.

Fix $x, y, m > 0$. Pick ε such that $f(x + y) - f(x + y - \varepsilon) < m$ and $f(\varepsilon) < m$. By claim 2 $f(x) + f(y) = f(\varepsilon) + f(x + y - \varepsilon) \Rightarrow |f(x + y) - f(x) - f(y)| < 2m, \forall m > 0 \Rightarrow f(x) + f(y) = f(x + y)$.

It is well-known that if $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is additive then there exists c such that $f(x) = cx$. But now it is easy to see that $c = 1$, so $f(x) = x$, $\forall x > 0$.

Solution 2: Quite clearly, the function is increasing. Let $\lim_{x \to 0^+} f(x) = l \geq 0$. By taking $x_i \to 0^+$ for all i, we get $l^2 = nl^2 + n(n - 1)l$, implying $l = 0$. Fixing $x_1 = x_0$ and letting $x_i \to 0^+$ for all $i > 1$ gives

$$\lim_{x \to x_0^+} f(x) = f(x_0),$$

while fixing $x_1 + x_2 + \cdots + x_n = x_0$ and letting $x_1 \to x_0^-$ gives

$$\lim_{x \to x_0^-} f(x) = f(x_0),$$

therefore, f is continuous.

Now letting $x_i \to 0^+$ for all $i > 2$ we arrive at the functional equation for $n = 2$, namely

$$f^2(x + y) = f^2(x) + f^2(y) + 2f(xy).$$

Substituting $y \to y + z$ and applying the equation again gives

$$f^2(x + y + z) = f^2(x) + f^2(y) + f^2(z) + 2(f(xy + xz) + f(yz)).$$

Exploiting symmetry we arrive at

$$f(xy + xz) + f(yz) = f(yx + yz) + f(xz),$$

implying that f satisfies the Jensen functional equation, from which we conclude that $f(x) = x$.

Problem 3. Let ABC be a scalene acute triangle with incenter I and circumcircle Ω. M is the midpoint of small arc BC on Ω and N is the projection of I onto the line passing through the midpoints of AB and AC. A circle ω with center Q is internally tangent to Ω at A, and touches segment BC. If the circle with diameter IM meets Ω again at J, prove that JI bisects $\angle QJN$.

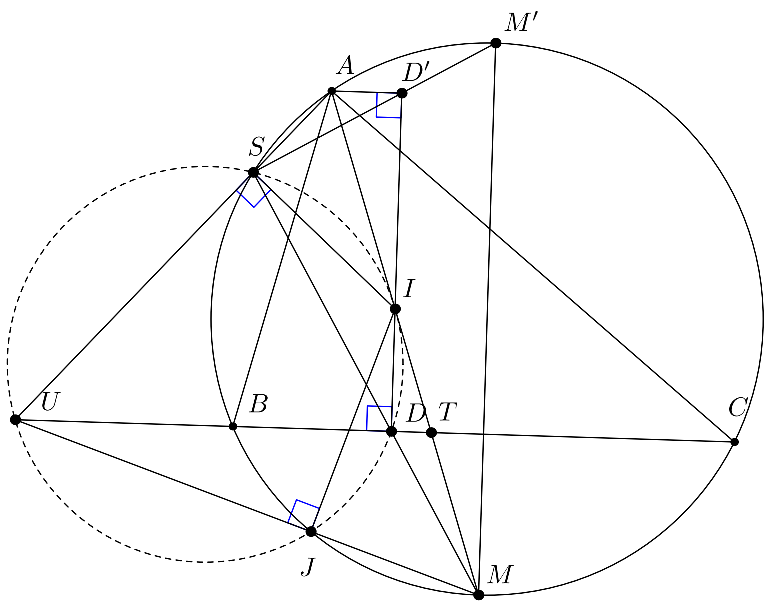
Anghel David-Andrei

Solution: Through the solution we will repeatedly use the following known fact: if a line through M meets BC and Ω at G and G', then $MG \cdot MG' = MI^2$.

Let D be the projection of I onto BC and denote the circle with center N and passing through D by τ. Let MJ and BC intersect again at U.

Main claim: U is the exsimilicenter of τ and ω.

Proof: Let D be the projection of I onto BC, and let MD meet Ω again at S. By our observation, $MD \cdot MS = MJ \cdot MU = MI^2$, so points S, D, J, U lie on a circle. But points I, U, D, J also lie on the circle with diameter IU, thus these 5 points lie on the circle with diameter UI. Now, let AI and BC meet at T.

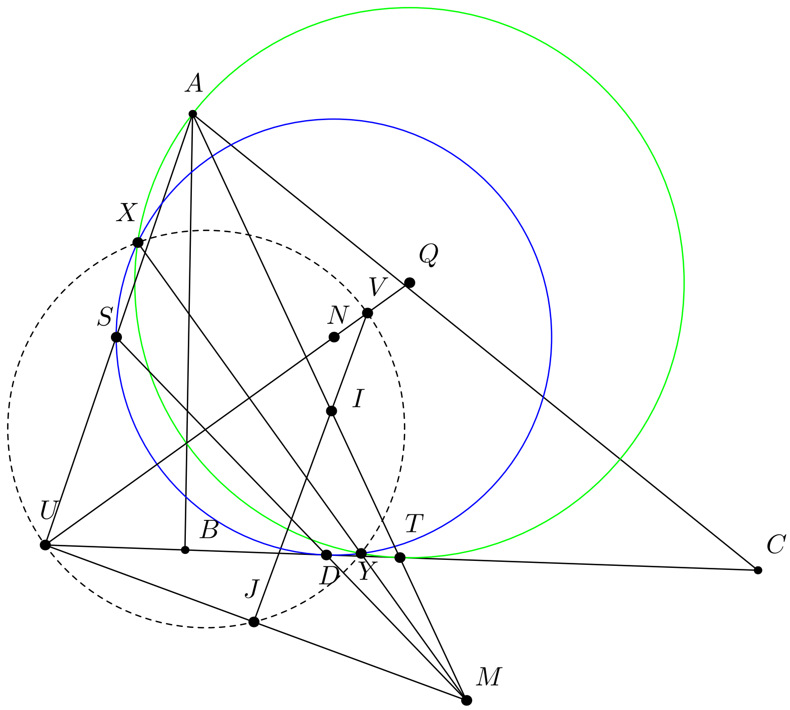


Since $MI^2 = MS \cdot MD = MA \cdot MT$, the first implies $\triangle MID \sim \triangle MSI$, so $\angle ISM = \angle DIM$, and the second implies $ASDT$ is cyclic, so $\angle DSA = \angle DTM$. Combining these, we get $\angle ASI = \angle IDT = 90°$, implying that A, S, U are collinear, as $IS \perp SU$.

Now let D' be the reflection of D over N, so that τ is the circle with diameter DD', and D' will be the projection of A on ID, so A, S, D', I lie on the circle with diameter AI, implying that $\angle DSD' = \angle DSA - \angle D'SA = \angle DTM - \angle D'IA = 90°$, so S lies on τ.

Furthermore, if we let SD' meet Ω again at M', observe that $SM' \perp SM$, so Ω is the circle with diameter MM', but then by homothety at S, τ and Ω are tangent at S. As D lies on segment BC, it will also lie on segment SM, so τ lies inside Ω.

But by the previous argument, S is the exsimilicenter of τ and Ω, and we know that A is the exsimilicenter of ω and Ω, thus, by Monge's theorem, AS passes through the exsimilicenter of τ and ω. But U lies on both this line and line BC, which is externally tangent to both Ω and ω, so the claim is proven.



Let V be the insimilicenter of τ and ω. It's known and easy to prove by homothety at A that T lies on ω, so I lies inside both τ and ω, A lies outside τ but on ω and D lies inside on τ but outside ω, so these two circles have 2 common points X and Y.

Observe that $MA \cdot MT = MS \cdot MD$, so M lies on the radical axis of τ and ω. Also, $MX \cdot MY = MD \cdot MS = MU \cdot MJ$, so X, Y, J, U lie on a circle. But observe that XU and XV are the 2 bisectors of $\angle NXQ$, from U and V being exsimilicenter and insimilicenter, so X lies on the circle with diameter UV, and similarly so does Y.

Thus, J also lies on this circle, so $VJ \perp JU$, but also $IJ \perp JU$, thus I lies on JV. But now, $(N, Q; V, U)$ make a harmonic bundle and $JV \perp JU$, so JV is the bisector of $\angle NJQ$, and we are done.