The Golden Digits National Contest

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Problem 1. Let $n > 1$ be a composite integer and $d_1 < \cdots < d_m$ be all its positive divisors. Is it possible for $d_i + d_{i+1}$ to be a perfect k-th power, $k \geqslant 2$ being fixed, for every $1 \leqslant i < m$?

Pavel Ciurea

Solution. The answer is no. Suppose, for the sake of contradiction, that there exist n and k with the desired properties. Observe that $d_1 = 1$ and $d_2 = p$, the smallest prime factor of n.

Because n is composite, $m \geqslant 3$. Therefore d_3 exists and is either equal to p^2 or some other prime number q. The former is impossible, because $p + 1$ and $p(p + 1)$ cannot simultaneously be perfect k-th powers, so $d_3 = q$, where q is another prime number.

We may now infer that $p + 1$ and $p + q$ are perfect k-th powers. Because $d_m = n, d_{m-1} = n/p$ and $d_{m-2} = n/q$ it also follows that $n(p + 1)/p$ and $n(p + q)/(pq)$ are perfect k-th powers. This being said, n/p and $n/(pq)$ must simultaneously be perfect k-th powers, which is absurd.

Remark. If one allows n to be prime as well, it can easily be observed that the only solutions are the prime numbers of the form $n = 2^p - 1$, where p itself is prime. These numbers are called *Mersenne primes* and it is unknown whether infinitely many of them exist.

Problem 2. We are given an infinite set of points in the plane such that any two of them have a distance of at most one. Prove that all the axes of symmetry of this set are concurrent, provided that there are at least two of them.

Note: An axis of symmetry of a set of points in the plane is a line ℓ with the property that the reflection of any point in the set with respect to ℓ is also in the set.

David-Andrei Anghel

Solution. Let ℓ_1 and ℓ_2 be two axes of symmetry. First, we prove that they are not parallel.

Assume the contrary, so ℓ_1 and ℓ_2 are parallel. Suppose further that they are also parallel to the x-axis, having y-coordinates a and b. Thus, if a point $P_0 = (c, d)$ is in the set, then its reflection P_1 in ℓ_1 belongs to the set and the reflection P_2 of P_1 in ℓ_2 also belongs to the set.

Observe that $P_2 = (c, 2(a - b) + d)$ and $a \neq b$. Thus, we can continuously perform this procedure, so for any positive integer n, the point $P_{2n} = (c, 2n(a - b) + d)$ belongs to the set. This obviously yields a contradiction, because of the distance condition.

To conclude, ℓ_1 and ℓ_2 meet at some point P. Consider some other axis of symmetry ℓ and assume that it does not pass through P. Let $d = \operatorname{dist}(P, \ell)$ and consider an arbitrary point X from the set. We may now prove the claim at the core of the solution.

Claim. There exists a point Y in the set, such that $PY^2 \geqslant PX^2 + 4d^2$.

Proof. Let ℓ' be the parallel to ℓ passing through P. A reflection in ℓ_1 combined with a reflection in ℓ_2 is a rotation with the pivot in P. By considering the reverse procedure (that is, reflecting in ℓ_2 and then ℓ_1), we may assume that the angle of this rotation is at most π.

Evidently, the angle of rotation is non-zero, because $\ell_1 \neq \ell_2$. Therefore, we can repeatedly apply this rotation on X until we get a point X' which satisfies $PX = PX'$ and which is not on the same side of ℓ' as ℓ. Take Y to be the reflection of X' in ℓ.

Note that $X'Y$ is perpendicular to ℓ. Let $Z = X'Y \cap \ell$. By Pythagoras' theorem, we have

$$PY^2 - PX'^2 = PZ^2 + YZ^2 - PZ^2 - X'Z^2 = (YZ - X'Z)(YZ + X'Z).$$

As $YZ - X'Z = 2d$ and $X'Z \geqslant 0$, we get the desired inequality. □

To see why the claim finishes the problem, take an arbitrary point A_0 from the set, and use the claim inductively to prove the existence of a point A_n from the set, such that $PA_n^2 \geqslant PA_0^2 + 4nd^2$. Therefore, $1 + PA_0 \geqslant A_0A_n + PA_0 \geqslant PA_n \geqslant 2d\sqrt{n}$, a contradiction for large enough n.

Remark. If there are at least three axes of symmetry, one may give an alternative solution which makes use of the following property: given a triangle Δ, for an adequate ordering a, b, c of its sides, the series of reflections in a, b, c, a, b, c produces a non-trivial translation.

The proof consists of computations with the aid of complex numbers, so we will not present it.

Problem 3. On the surface of a sphere, a non-intersecting closed curve comprised of finitely many circle arcs is drawn. It divides the surface of the sphere in two regions, which are coloured red and blue. Prove that there exist two antipodes of different colours (the curve is colourless).

Vlad-Titus Spătaru

Solution 1. Let $\mathcal{C}$ be the curve on the sphere's surface. Of course, note that the red and blue regions are connected. Suppose, for the sake of contradiction, that there is no pair of antipodes with the desired property. Let $\mathcal{R}$ be the red region.

Consider any red point p_1 on the sphere, whose antipode p_2 does not lie on $\mathcal{C}$. It follows from our assumption that p_2 is red. Call a path *circular* if it consists of finitely many circle arcs. Define analogously a circular curve.

Claim 1. There exists an entirely red circular path Π_r from p_1 to p_2.

Proof. Firstly, consider some arcs p_1q_1 and p_2q_2 where q_1, q_2 lie on $\mathcal{C}$. Consider the circular path $\Pi \subset \mathcal{C}$ which connects q_1 and q_2. Note that because $\mathcal{C}$ is comprised of *finitely many* circle arcs, there exists a constant $\varepsilon > 0$ so that $\operatorname{dist}(a, b) > \varepsilon$ for any points a, b on the $\mathcal{C}$.

Therefore, the path Π_0 may be translated by a small enough ε' such that, letting $q_1 \mapsto \tilde{q}_1, q_2 \mapsto \tilde{q}_2$ and $\Pi \mapsto \tilde{\Pi}$, the circular path $\Pi_r := p_1\tilde{q}_1 \cup \tilde{\Pi} \cup \tilde{q}_2p_2$ lies entirely in the red region $\mathcal{R}$. □

Claim 2. Let Π'_r be the reflection of Π_r about the sphere's centre. Then, Π'_r does not intersect $\mathcal{C}$.

Proof. Assume that a point $q \in \Pi'_r$ belongs to $\mathcal{C}$ and let $q' \in \Pi_r$ be the antipode of q. Consider a small enough neighbourhood $\mathcal{D}$ of q' which is entirely red. Consider any point $\tilde{q} \in \mathcal{D}$.

Let $\tilde{q}'$ be the antipode of $\tilde{q}$ and $\mathcal{D}'$ be the reflection of $\mathcal{D}$ about the sphere's centre. Of course, $\mathcal{D}'$ is a neighbourhood of q. It follows from our assumption that $\tilde{q}'$ is red, or it belongs to $\mathcal{C}$ so, generalizing, every point in $\mathcal{D}'$ is red or belongs to $\mathcal{C}$.

Since $q' \in \mathcal{C}$, then $\mathcal{C}$ pierces the neighbourhood $\mathcal{D}'$ at least once, splitting it into several regions. At least two of these regions lie on different sides of $\mathcal{C}$, so one of them must be blue, which contradicts the former observation. Therefore, $\Pi'_r \cap \mathcal{C}$ is empty. □

Claim 3. Provided that Π_r is adequately chosen, the path Π'_r does not intersect the path Π_r.

Proof. Let $\alpha_1, \ldots, \alpha_n$ be the arcs that make up Π_r and for each $i = 1, \ldots, n$ let ω_i be the circle containing α_i. Note that we may assume that $\omega_i \neq \omega_j$ for all i, j, for otherwise we can translate some arc $\alpha_i \mapsto \tilde{\alpha}_i$ by a small enough amount ensuring $\omega_i \mapsto \tilde{\omega}_i \neq \omega_j$ for any j.

Now, observe that any intersection between Π_r and Π'_r is an intersection between some circles ω_i, ω_j and of course, there are finitely many such intersection points. Also, observe that if $q \in \Pi_r \cap \Pi'_r$ then the antipode q' of q also belongs to $\Pi_r \cap \Pi'_r$. If Π_r and Π'_r do not have any intersections, we are done. Otherwise, let $(q_1, q'_1), \ldots, (q_k, q'_k)$ be all the intersection points of Π_r and Π'_r.

For each $i = 1, \ldots, k$ let $\Pi_{r,i} \subset \Pi_r$ be the path from q_i to q'_i. Evidently, all these paths have finite lengths. Thus, we may choose some index m such that $\Pi_{r,m}$ has minimal length, ties broken arbitrarily. Assume, for the sake of contradiction, that $\Pi_{r,m}$ and $\Pi'_{r,m}$ intersect each other.

Consider a point $q \in \Pi_{r,m} \cap \Pi'_{r,m}$. As noted above, the antipode q' of q must also be an intersection point of $\Pi_{r,m}$ and $\Pi'_{r,m}$, so $(q, q') = (q_i, q'_i)$ for some index i. Thus, $\Pi_{r,i} \subset \Pi_{r,m}$, contradicting the minimality of the length of $\Pi_{r,m}$.

To conclude, the circular paths $\Pi_{r,m}$ and $\Pi'_{r,m}$ do not intersect each other. Therefore, we may assume that Π_r was originally chosen as $\Pi_{r,m}$, proving the claim. □

By combining the claims with the definition of Π'_r it follows that $\Pi_r \cup \Pi'_r$ is a red, non-intersecting closed curve circular $\mathcal{C}_r$ symmetrical with respect to the sphere's centre. Then, $\mathcal{C}_r$ divides the surface of the sphere into two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ of equal area.

Since the blue region is connected, then there cannot be blue points in both $\mathcal{R}_2$ and $\mathcal{R}_2$, for otherwise a blue path must cross the red curve $\mathcal{C}_r$. Hence, some region $\mathcal{R}_i$ is entirely red.

The boundary of $\mathcal{R}_i$ is the red curve $\mathcal{C}_r$ while the boundary of $\mathcal{R}$ is the curve $\mathcal{C}$, which does not intersect $\mathcal{C}_r$. Therefore, the area of $\mathcal{R}$ is strictly greater than the area of $\mathcal{R}_i$, which is half of the sphere's surface area S.

However, by analogous arguments, we may infer that the area of the blue region $\mathcal{B}$ is strictly greater than half of the sphere's surface area as well, which leads to a contradiction, since the areas of $\mathcal{R}$ and $\mathcal{B}$ add up to S.

Remark 1. One may convert the problem to the two-dimensional plane via an adequately chosen *stereographic projection* of the sphere's surface. We will sketch the solution for this approach.

The curve $\mathcal{C}$ is mapped to a circular non-intersecting curve $\tilde{\mathcal{C}}$ in the plane and the red and blue region are mapped to the interior and exterior of $\tilde{\mathcal{C}}$, depending on the choice of the poles of the stereographic projection. Of course, there is no difference.

Letting s be the south pole of the projection, the pre-images of two points p_1, p_2 in the plane are antipodes if and only if s lies on the segment p_1p_2 and $\text{dist}(s, p_1) \cdot \text{dist}(s, p_2) = \lambda$ for some constant λ, that is, p_1, p_2 are switched by an inversion in s composed with a reflection in s.

For the sake of brevity, let $\gamma : \mathbb{R}^2 \setminus \{s\} \to \mathbb{R}^2 \setminus \{s\}$ be the geometric transformation we have just described, which is an involution. The there claims and their proofs are isomorphic in this case, except that instead of antipodes we refer to pairs of points $p, \gamma(p)$.

Assume, for the sake of contradiction, that there is no pair of points $p, \gamma(p)$ of different colours. Without loss of generality, assume further that the south pole s is coloured blue. Now, choose a red point p for which $\gamma(p)$ is also red, which of course exists.

From the three claims, there exists a red circular path Π_r from p to $\gamma(p)$, for which $\gamma(\Pi_r)$ is also red and does not intersect Π_r. Then, Π_r and $\gamma(\Pi_r)$ form a red non-intersecting circular curve, with the point s lying in its interior.

Since the blue region is connected, there is no blue point outside of the curve $\mathcal{C}_r = \Pi_r \cup \gamma(\Pi_r)$. Therefore, considering a blue point s' which is close enough to s, due to the nature of γ, the point $\gamma(s')$ will lie outside $\mathcal{C}_r$, so it must be red, which finishes the proof.

Remark 2. We are only working with non-intersecting circular curves, so there is no ambiguity regarding the existence of an interior and exterior of the curve. Should we consider an arbitrary non-intersecting closed curve $\mathcal{C}$, multiple problems arise.

We will now present a non-elementary solution, for an arbitrary non-intersecting closed curve $\mathcal{C}$.

Solution 2. Using Jordan's theorem, the red and blue regions exist. Just like before, assume for the sake of contradiction that there is no pair of antipodes with the desired property. The core of the solution is the alternative definition of compactness in $\mathbb{R}^n$.

Theorem (Heine-Borel)**.** A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Let $\mathcal{R}$ be the set of red points, which is open. We may now prove the following, crucial claim.

Claim 1. For any red points p_1 and p_2, there exists a red path Π_r of finite length from p_1 to p_2.

Proof. The red region is connected, so we may choose an arbitrary red path Π from p_1 to p_2. For every point $q \in \Pi$, consider an open ball B_q containing q for which $B_q \cap \mathbb{S}^2 \subseteq \mathcal{R}$. From the Heine-Borel theorem, observe that $\Pi \subset \mathbb{R}^3$ is a compact set.

Thus, the open cover $\{B_q : q \in \Pi\} \supseteq \Pi$ contains some finite open sub-cover $\{B_k : 1 \leqslant k \leqslant n\} \supseteq \Pi$. Consider the open spherical caps $C_k = B_k \cap \mathbb{S}^2$. Evidently, Π belongs to the union of these caps. Thus, there exists a path Π_r of finite length from p_1 to p_2, comprised of arcs of these caps. □

The solution continues naturally. Let p_1 and p_2 be two red antipodes and let Π_r be a red path of finite length connecting the two. We prove claim 2 analogously, but claim 3 requires more care.

Claim 2. Let Π_r' be the reflection of Π_r about the sphere's centre. Then, Π_r' does not intersect $\mathcal{C}$.

Claim 3. Provided that Π_r is adequately chosen, the path Π_r' does not intersect the path Π_r.

Proof. Call a red path *good* if it connects two red antipodes and has finite length. Let $\mathcal{P}$ be the collection of good paths $\Pi \subseteq \Pi_r$. For any path Π, let $\ell(\Pi)$ be its length.

Take $\mu = \inf \ell(\Pi)$ as $\Pi \in \mathcal{P}$. Observe that any path between two antipodes has length at least πr, where r is the sphere's radius, so $\mu > 0$. Now, consider a sequence of good paths $(\Pi_n)_{n\geqslant 1}$ for which the lengths $\ell(\Pi_n)$ tend to μ as n tends to infinity.

For each path $\Pi \in \mathcal{P}$ let $p_1(\Pi)$ be the endpoint of Π which is closer to p_1 and $p_2(\Pi)$ be the endpoint of Π which is closer to p_2. By using the Bolzano-Weierstrass theorem on the sequence $p_1(\Pi_n)_{n\geqslant 1}$ extract the convergent sequence $p_1(\Pi_{n_i})_{i\geqslant 1}$.

Since $\ell(\Pi_n)$ tends to μ, then so does $\ell(\Pi_{n_i})$. Consequently, the sequence $p_2(\Pi_{n_i})$ is convergent as well. We may the infer that the paths Π_{n_i} tend to a path of length μ which we will denote by Π_∞. Also, because Π_{n_i} are all good paths, then Π_∞ is also a good path.

We claim that Π_∞ and Π_∞' do not intersect each other. Assume, for the sake of contradiction, that some point q belongs to both of these paths. Since Π_∞ and Π_∞' are symmetrical, then they also intersect at q', the antipode of q.

Then, the path $\Pi \subset \Pi_\infty$ which connects q and q' would also be good, and $\ell(\Pi) < \mu$ which is a contradiction. Therefore, Π_∞ and Π_∞' do not intersect each other, so assuming we chose Π_r to be Π_∞ beforehand, the claim holds. □

We may now finish analogously to solution 1.