The Golden Digits National Contest

4th Edition, May 2024

Problem 1. Let $n \geqslant 2$ be an integer. Prove that for any positive real numbers $a_1, a_2, \ldots, a_n$,

$$\frac{1}{2\sqrt{2}} \sum_{i=1}^{n} 2^i a_i^2 \geqslant \sum_{1 \leqslant i < j \leqslant n} a_i a_j.$$

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Solution. We prove the inequality by inducting on n. For $n = 2$, the base case, the inequality is equivalent to $a_1^2 + 2a_2^2 \geqslant \sqrt{2} a_1 a_2$ which is true by the am-gm inequality. Now, assume that the inequality is true for some integer $n \geqslant 2$. We will prove it also holds for $n + 1$.

By applying the inequality for the n terms $a_2, \ldots, a_{n+1}$ it follows that

$$\frac{1}{2\sqrt{2}} \sum_{i=2}^{n+1} 2^{i-1} a_i^2 \geqslant \sum_{2 \leqslant i < j \leqslant n+1} a_i a_j.$$

Therefore, in order to prove the general inequality for $n + 1$, it suffices to show that

$$\frac{1}{2\sqrt{2}} \left(2a_1^2 + \sum_{i=2}^{n+1} 2^{i-1} a_i^2 \right) \geqslant a_1(a_2 + \cdots + a_{n+1}). \tag{†}$$

One may easily observe that $2 \geqslant 2^0 + 2^{-1} + \cdots + 2^{-(n-1)}$. Inequality (†) then follows as such:

$$\frac{1}{2\sqrt{2}} \left(2a_1^2 + \sum_{i=2}^{n+1} 2^{i-1} a_i^2 \right) \geqslant \sum_{i=2}^{n+1} \frac{2^{-(i-2)} a_1^2 + 2^{i-1} a_i^2}{2\sqrt{2}} \geqslant \sum_{i=2}^{n+1} a_1 a_i,$$

by applying the am-gm inequality on each term.

Remark 1. More generally, with the same approach, one can prove that

$$y \sum_{i=1}^{n} x^i a_i^2 \geqslant \sum_{1 \leqslant i < j \leqslant n} a_i a_j,$$

provided that $x > 1$ and $2y(x - 1) \geqslant 1/\sqrt{x}$. In the problem above, we used $x = 2$ and $y = 1/2\sqrt{2}$.

Remark 2. We present another possible approach. The inequality is equivalent to

$$\sum_{i=1}^{n} \left(1 + \frac{2^i}{\sqrt{2}} \right) a_i^2 \geqslant \left(\sum_{i=1}^{n} a_i \right)^2,$$

which is true by the Cauchy-Schwarz inequality, because

$$\sum_{i=1}^{n} \frac{1}{1 + 2^i/\sqrt{2}} < \sum_{i=1}^{\infty} \frac{1}{1 + 2^i/\sqrt{2}} \approx 0.9927 < 1.$$

Problem 2. Let n be a positive integer. Consider an infinite checkered board. A set S of cells is *connected* if one may get from any cell in S to any other cell in S by only traversing edge-adjacent cells in S. Find the largest integer k_n with the following property: in any connected set with n cells, one can find k_n disjoint dominoes.

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Solution. The answer is $k_n = \lfloor (n+2)/4 \rfloor$. Firstly, we show that we can always find $\lfloor (n+2)/4 \rfloor$ disjoint dominoes in a connected set with n cells. To do so, we will reinterpret the problem using a graph. Consider an arbitrary connected set S with n cells.

Construct a graph G, where each vertex represents a cell in our connected set and two vertices are connected if and only if their corresponding cells share an edge. Naturally, finding k_n disjoint dominoes is equivalent to finding k_n disjoint edges.

Because any cell is adjacent to four other cells of the board, for any vertex $v \in G$ we have $\deg v \leqslant 4$. Consequently, it suffices to prove the following graph-theoretic claim:

Claim. In any connected graph G with n vertices and maximal degree $\Delta \leqslant 4$, there exist at least $k_n = \lfloor (n+2)/4 \rfloor$ disjoint edges.

Proof. We will prove this by strong induction on n. The cases $n \leqslant 5$ are trivially true. For $n \geqslant 6$, assume that the claim holds for $1, 2, \ldots, n-1$. We will show that it also holds for n.

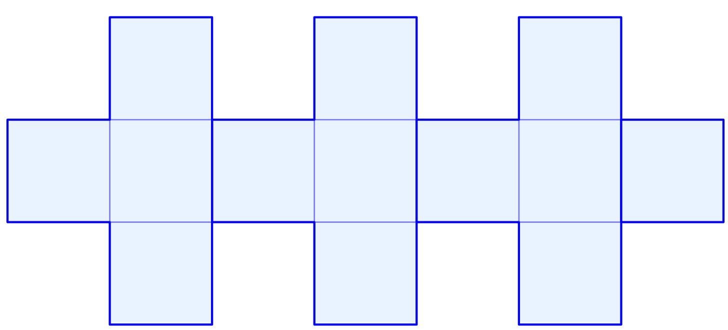
This graph is connected, so we may consider a spanning tree T of G. Consider an arbitrary vertex v_0 as the root of T. As $|T| \geqslant 6$ then there exists at least one vertex v so that $\operatorname{dist}(v_0, v) \geqslant 2$. Hence, let $\max_{v \in T} \operatorname{dist}(v_0, v) = d \geqslant 2$.

Consider a vertex u so that $\operatorname{dist}(v_0, u) = d$. Let w be the parent of u and L_w be the set of leaves emerging from w. As $\operatorname{dist}(v_0, w) = d - 1 \geqslant 1$, then w also has a parent. Since $\deg w \leqslant 4$ then $|L_w| \leqslant 3$. Consider the edge uw and remove $L_w \cup \{w\}$ from T.

We thus form a tree T' with $|T| - 1 - |L_w| \geqslant n - 4$ vertices. Hence, by the inductive hypothesis, in this tree exclusively we may find $\lfloor (|T'|+2)/4 \rfloor \geqslant \lfloor (n+2)/4 \rfloor - 1$ disjoint edges. By also counting uw, we get the desired $\lfloor (n+2)/4 \rfloor$ disjoint edges in T, finishing the proof. □

Now, it suffices to provide an example for which no more than $k_n = \lfloor (n+2)/4 \rfloor$ dominoes can be selected. Evidently, it suffices to provide an example for $n = 4\ell + 1$, for which $k_n = \ell$, because for $n = 4\ell + \varepsilon$ with $\varepsilon \in \{0, -1, -2\}$, it suffices to remove some cells.

For $n = 4\ell + 1$, consider a sequence of ℓ T-tetrominoes, followed by a single cell, as seen below.



Problem 3. Let p be a prime number and $\mathcal{A}$ be a finite set of integers, with at least p^k elements. Denote by N_{even} the number of subsets of $\mathcal{A}$ with even cardinality and sum of elements divisible by p^k. Define N_{odd} similarly. Prove that $N_{\text{even}} \equiv N_{\text{odd}} \bmod p$.

Solution. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ with $n \geqslant p^k$. We will encode each subset of $\mathcal{A}$ as an n-digit code $(\chi_1, \ldots, \chi_n)$ where $\chi_i = 1$ if a_i belongs to the subset and $\chi_i = 0$ otherwise. Evidently, the sum of elements of the subset is $\chi_1 a_1 + \cdots + \chi_n a_n$.

We will construct an n-variable polynomial $f(x_1, \ldots, x_n)$ with integer coefficients, whose degree does not exceed $p^k - 1$ and which satisfies

$$f(\chi_1, \ldots, \chi_n) \equiv \begin{cases} 1 \bmod p & \text{if } p^k \mid \chi_1 a_1 + \cdots + \chi_n a_n; \\ 0 \bmod p & \text{otherwise} \end{cases} \tag{$\dagger$}$$

for any $\chi_1, \ldots, \chi_n \in \{0, 1\}$. To exhibit a construction, we will consider a sequence of monomials. The term x_1 is written a_1 times, followed by x_2 written a_2 times and so on, ending with x_n written a_n times and the constant 1 written $p^k - 1$ times:

$$(\underbrace{x_1, \ldots, x_1}_{a_1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{a_2 \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{a_n \text{ times}}, \underbrace{1, \ldots, 1}_{p^k - 1 \text{ times}}),$$

for a total of $N := a_1 + \cdots + a_n + p^k - 1$ monomials. Then, for each $1 \leqslant i \leqslant N$, let $g_i(x_1, \ldots, x_n)$ be the i-th term of this monomial sequence. Consider every $(p^k - 1)$-tuple of polynomials of the form g_i, take their product and sum everything in order to get

$$f(x_1, \ldots, x_n) := \sum g_{i_1}(x_1, \ldots, x_n) \cdots g_{i_{p^k - 1}}(x_1, \ldots, x_n),$$

for every $1 \leqslant i_1 < \cdots < i_{p^k - 1} \leqslant N$. Evidently, this polynomial has degree $d \leqslant p^k - 1$. Moreover, given any $\chi_1, \ldots, \chi_n \in \{0, 1\}$, then there are precisely $\chi_1 a_1 + \cdots + \chi_n a_n + p^k - 1$ ones among the numbers $g_i(\chi_1, \ldots, \chi_n)$ so

$$f(\chi_1, \ldots, \chi_n) \equiv \binom{\chi_1 a_1 + \cdots + \chi_n a_n + p^k - 1}{p^k - 1} \bmod p,$$

which satisfies the congruence ($\dagger$) due to Lucas' theorem. This construction is satisfactory.

Let $C_{u_1, \ldots, u_n}$ be the coefficient of the term $x_1^{u_1} \cdots x_n^{u_n}$ in the expansion of f. For the sake of brevity, in what follows, any instance of χ_i will denote a number in $\{0, 1\}$. The main observation is

$$N_{\text{even}} - N_{\text{odd}} \equiv \sum_{\chi_1, \ldots, \chi_n} (-1)^{\chi_1 + \cdots + \chi_n} f(\chi_1, \ldots, \chi_n) \bmod p.$$

We will denote $\chi_1 + \cdots + \chi_n$ by $\Sigma \chi_i$. Observe that the right-hand side is equal to

$$\begin{aligned} \sum_{\chi_1, \ldots, \chi_n} (-1)^{\Sigma \chi_i} \sum_{u_1, \ldots, u_n} C_{u_1, \ldots, u_n} \chi_1^{u_1} \cdots \chi_n^{u_n} &= \sum_{u_1, \ldots, u_n} C_{u_1, \ldots, u_n} \sum_{\chi_1, \ldots, \chi_n} (-1)^{\Sigma \chi_i} \chi_1^{u_1} \cdots \chi_n^{u_n} \\ &= \sum_{u_1, \ldots, u_n} C_{u_1, \ldots, u_n} \prod_{i=1}^{n} \left(\sum_{\chi_i \in \{0,1\}} (-1)^{\chi_i} \chi_i^{u_i} \right). \end{aligned}$$

Because $\deg f < p^k \leqslant n$ then among every $u_1, \ldots, u_n$ there exists at least one index i such that $u_i = 0$, hence the right-hand side is equal to zero, as desired.