The Golden Digits International Contest

5th Edition, October 2024

Problem 1. Find all functions $f : \mathbb{Z}_{>}0 \to \mathbb{Z}_{>}0$ with the following properties:

1) For every natural number $n \geq 3$, $\gcd(f(n), n) \neq 1$, where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

2) For every natural number $n \geq 3$, there exists $i_n \in \mathbb{N}$, $1 \leq i_n \leq n-1$, such that $f(n) = f(i_n) + f(n - i_n)$.

Ciurea Pavel

Solution: We show by complete induction on n that $\forall n$ we can find nonnegative integers u_n and v_n such that $f(n) = u_n f(1) + v_n f(2)$, and $u_n + 2v_n = n$.

The statement for $n = 1$ and $n = 2$ trivially holds. Suppose that the statement is true for 1, 2, $\ldots n-1$. Setting $u_n = u_{i_n} + u_{n-i_n}$, and $v_n = v_{i_n} + v_{n-i_n}$ we obtain $f(n) = u_n f(1) + v_n f(2)$ and $u_n + 2v_n = n$, so the claim in true for n as well, thus the induction is complete.

Now $\gcd(f(n), n) = \gcd(u_n f(1) + v_n f(2), n) = \gcd(nf(1) - 2v_n f(1) + v_n f(2), n) = \gcd(v_n(f(2) - 2f(1)), n)$. Since $u_n + 2v_n = n \Rightarrow v_n \leq n/2$, so by letting n be a prime number we get that $\gcd(f(2) - 2f(1), n) \neq 1$, so $p \mid f(2) - 2f(1)$, for any prime number p, thus $f(2) = 2f(1)$, so $f(n) = nf(1)$, $\forall n$. All such functions clearly work, so the proof is complete.

Problem 2. Let ABC be a triangle and P a point in its interior. Circle Γ_A is considered such that it is tangent to rays $(PB$ and $(PC$. Define similarly Γ_B and Γ_C. Prove that the other common internal tangents of the circles are concurrent in a point.

Andrei Vila

Solution: Forget points A, B, C and relabel the intersection of the common internal tangents of Γ_B and Γ_C by A. Define B and C similarly. Let the centers of $\Gamma_A, \Gamma_B, \Gamma_C$ be X, Y, Z. Let $\{Q\} = \ell_B \cap \ell_C$. We want to prove that QA is also tangent to Γ_B and Γ_C.

Suppose not and let QD be tangent to Γ_B such that $D \in PA$. Thus $BPCQ$ and $QPCD$ both have an exscribed circle (namely Γ_B), implying that

$$BP + BQ = CP + CQ \text{ and } CP + CQ = DP + DQ.$$

From this we get that

$$BP + BQ = DP + DQ,$$

implying that the quadrilateral $DPBQ$ is exscribed. Since Γ_C is already tangent to three of the sides of this quadrilateral, so is to the fourth, implying that QD is also tangent to Γ_C, finally giving $A \equiv D$, exactly what we wanted.

Problem 3. Let $a_1 < a_2 \cdots < a_n$ be positive integers, with $n \geq 2$. An invisible frog lies on the real line, at a positive integer point. Initially, the hunter chooses a number k, and then, once every minute, he can check if the frog currently lies in one of k points of his choosing, after which the frog goes from its point x to one of the points $x + a_1, x + a_2 \dots x + a_n$. Based on the values of $a_1, a_2 \dots a_n$, what is the smallest value of k such that the hunter can guarantee to find the frog within a finite number of minutes, no matter where it initially started?

Anghel David

We will prove the minimum such k is $\frac{a_n - a_1}{gcd(a_2 - a_1, a_3 - a_1 \dots a_n - a_1)} + 1$. For this, we will first prove 3 claims:

Claim 1 : The hunter can find the frog if and only if for any positive integer M he can find the frog with the extra knowledge that its initial position is at most M.

Proof : Clearly this only advantages the hunter, so we only need to prove that if for any M he has a strategy S_M to find it with that extra knowledge, then he can find it in the initial problem. His strategy will be to treat the case when the initial position of the frog is $< 10^1$, when the starting position is $< 10^2$, and so on, and the initial position will eventually be considered in one of these cases.

When we get to the case i, let's say we've made t moves, then we know that if the initial position was at most 10^i, the current position after the first $i-1$ cases is at most $10^i + t \cdot a_n$, Thus we have a strategy in $S_{10^i + t \cdot a_n}$ steps and we are done.

Thus, we can suppose that we know a number M such that the initial position is at most M.

Claim 2 : We can reduce the problem to the case $a_1 = 0$

Proof : We consider our numbers to be $0, a_2 - a_1 \dots a_n - a_1$. Then checking position p after t minutes with the initial numbers is equivalent to checking position $p - ta_1$ with the new numbers, thus we find a bijection between the game before and the game after shifting the a_i's, so we can just consider the case $a_1 = 0$.

From now on, suppose that $a_1 = 0$, and let $d = gcd(a_2 \dots a_n)$

Claim 3 : The minimal value of k for $a_2 \dots a_n$ is the same as the minimal value of k for $\frac{a_2}{d} \dots \frac{a_n}{d}$

Proof : We call a number k *good* for a_2, a_3, $\dots a_n$ if the hunter can catch the frog by checking k points each minute. Suppose the hunter has the additional information that the frog starts on a point divisible by d. Since the length of each jump is divisible by d, the frog only jumps on multiples of d. But this game is obviously equivalent to the game where the frog can start anywhere and the a_i's are $\frac{a_2}{d} \dots \frac{a_n}{d}$. So if k is *good* for a_2, a_3, $\dots a_n$ k is also *good* for $\frac{a_2}{d} \dots \frac{a_n}{d}$. Suppose k is *good* for $\frac{a_2}{d} \dots \frac{a_n}{d}$, and that if the hunter knows the frog starts before a number M he can catch it after t_M minutes. Then if in the game where the a_i's are a_2, a_3, $\dots a_n$ if he knows the frog started before number $d \cdot M$ and on a number $\equiv r \pmod d$ he can find the frog after t_M minutes. Then if he doesn't catch the frog after t_M minutes, he knows the frog did not start on a number $\equiv r \pmod d$, and he can inductively eliminate residues until he catches the frog. Then, by claim 1, k is also *good* for a_2, a_3, $\dots a_n$.

Finally, we are left with proving that if the a_i are coprime nonnegative integers with 0 appearing in the sequence, and the biggest a_i is x, then the minimal k is $x + 1$.

It's trivial to see that $x+1$ is enough: if we know the position of the frog is at most N, the hunter chooses $N-x \dots N$, and if he doesn't find it he knows the frog's position is at most $N-x-1$, and thus the new position is at most $N-1$, and we can repeat this process until we get that it's position less than 0, contradiction.

Now we prove we need at least x. Suppose $k=x$ works. We prove that if the number of positions the frog can be in is at least x^3+1, then after each move this number doesn't decrease (we consider the possibilities left after the hunter eliminates x of them).

Let $f(i)$ be the number of possibilities that are equal to i modulo x. Then as 0 appears in our sequence, all of these remain as possibilities. If each $f(i)$ is non-zero we are done, as in each residue class mod x we still have the same possibilities and we also get the biggest one $+x$, as x is in our sequence, so we are done.

Thus, one $f(i)$ needs to be 0. Consider a graph with vertices $1, 2 \dots x$, and a directed edge from a to b if the residue of $b-a$ in mod x appears in the sequence a_i. Then, if for an edge $a \implies b$ we have $f(a) \geq f(b)+x$, we are done, as we can let $b-a \equiv a_j (mod x)$.

Thus when can add a_j to each possibility with residue class a and we get $f(a)$ residues in class b, out of each at most $f(b)$ were already there, thus we get at least x new residues. So, for each edge, we have $f(a) < f(b)+x$. But now, as the a_i are coprime, the graph is strongly connected.

Let v be a residue class with $f(v)=0$ and u a random residue class. Thus we have a directed path from u to v, but then the minimal path clearly has length at most x, and, at each step, the f of the residues class must decrease by less than x, thus we get that $f(u)$ is less than x^2, but when summing over all residue classes we get the number of possibilities is at most $x^2 \cdot x$, contradiction.

So, if we initially have x^3+1 possibilities, we will always have at least $x^3+1-x>0$ possibilities, which concludes the proof.